Integral Equations for Radiative Transfer with Linear Anisotropic Scattering and Fresnel Boundaries

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Transformation of the integro-differential transport equation in terms of radiation intensity to integral equations in terms of moments of the radiation intensity reduces the computational labor because the former depends on position and direction and the latter depends on position only. Our analysis deals with two cases for which the scattering is linearly anisotropic. One involves radiative transfer in an arbitrary three-dimensional medium with a given inward boundary intensity and the other involves radiative transfer in a three-dimensional rectangular medium with Fresnel boundaries and the top surface exposed to normal incidence. Because the inward boundary intensity is unknown in the second case, an image technique is used to generate integral equations similar to those obtained in the first case. Numerical results for specific examples are given. Comparing the results for a slab with nonreflecting boundaries with existing exact solutions shows that our analysis works quite well.

Nomenclature

a_1	= expansion coefficient in phase function
A	= area
h	= moment of the boundary intensity
I	= intensity
I_b	= blackbody intensity
I_c	= collimated incident intensity
$\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$	= unit vector
j	= moment of the medium intensity
n	= refractive index
$\boldsymbol{n}$	= unit normal vector
p	= number denoting the process of reflection (see Fig. 3)
P	= phase function
$\boldsymbol{r}$	= position vector
s	= path length
S	= source function
T	= temperature
V	= volume
x,y,z	= coordinate
β	= extinction coefficient
$\Delta\tau$	= distance
θ	= polar angle
θ_{cr}	= critical angle
κ	= absorption coefficient
μ	= $\cos\theta$
ρ	= reflectivity
ρ_n	= normal reflectivity
Σ	= summation
σ	= scattering coefficient
τ_x, τ_y, τ_z	= optical coordinates
ϕ	= azimuthal angle
ω	= scattering albedo
Ω	= solid angle
$\boldsymbol{\Omega}$	= unit vector along the path of a ray

Subscripts

i	= incidence
ℓ,m,n	= integration involving $(\boldsymbol{r}-\boldsymbol{r}')\cdot\boldsymbol{i}$, $(\boldsymbol{r}-\boldsymbol{r}')\cdot\boldsymbol{j}$, and $(\boldsymbol{r}-\boldsymbol{r}')\cdot\boldsymbol{k}$, respectively
p	= quantity in the pth image
x,y,z	= x,y,z components, respectively

Introduction

RADIATIVE transfer in anisotropically scattering media has been of interest for a long time.[1,2] The transport equation governing radiative transfer in a scattering medium is an integro-differential equation in terms of a radiation intensity. For the physical situation involving reflection at the boundaries, this equation is subject to boundary conditions in which the radiation intensity is unknown. The direct solution of this intensity formulation is extremely laborious, especially for multidimensional cases. Because the source function for the radiative transfer in isotropically scattering media is a function of spatial variables only, many authors[3-7] developed the integral equation of the source function to reduce the labor of computation. But, for the radiative transfer in anisotropically scattering media, the source function, as well as the radiation intensity, depends not only on position but also on direction; thus, formulating problems in terms of the source function is not especially advantageous. To circumvent this difficulty, Hunt[8] developed a technique to reduce the integro-differential transport equation to a set of integral equations in terms of moments of intensity. Since the moments of intensity are the functions of position only, Hunt's technique is superior to others. However, Hunt[8] derived integral formulation only for a specific problem with rectangular geometry and nonreflecting boundaries. Therefore, in the first part of the present analysis, Hunt's technique is simplified by the application of vector notation and is extended to derive the exact integral equations for the radiative transfer in an arbitrary three-dimensional medium with given inward boundary intensity.

In most published analyses for the multidimensional radiative transfer in anisotropically scattering media, reflecting boundaries have not been accounted for. However, in many instances, the problem of interest will include the effects of partially reflecting boundaries. Therefore, in the second part of the present analysis, the effects of Fresnel boundaries whose reflectivities are directionally dependent are taken into account for a three-dimensional rectangular geometry. Because the inward intensity of radiation at Fresnel boundaries is not given, the image technique developed by Wu, Sutton, and Love[6] for the radiative transfer in isotropically scattering media is adapted for the present anisotropically scattering case. The

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results of the second part show that the number of the integral equations or their unknowns does not change. That is, the image technique is still valid for the radiative transfer in anisotropically scattering media with Fresnel boundaries. Because only one-dimensional cases for anisotropic scattering have analytical exact solutions,[9] numerical computations are performed for those cases to give an initial evaluation of the present formulation.

For simplicity, the linearly anisotropic scattering case is considered here. The ideas presented here should still be applicable for more realistic anisotropic scattering.

Analysis

Integral Equations for General Three-Dimensional Medium

The basic assumptions about the system in which the radiative transfer takes place are: 1) the medium is emitting, absorbing, and anisotropically scattering; 2) the medium is in local thermodynamic equilibrium; 3) steady state is achieved; 4) the medium is homogeneous; 5) the scattered radiation has the same frequency as the incident radiation; 6) the geometrical dimension of the medium is much greater than one wavelength; and 7) the radiation intensity leaving the boundary is given. The subscript that denotes the spectrally dependent properties of the medium and the boundaries is omitted to simplify the mathematical expressions. The equation of radiative transfer may be expressed as

$$\frac{1}{\beta}\frac{\mathrm{d}I(s,\mathbf{\Omega})}{\mathrm{d}s} + I(s,\mathbf{\Omega}) = S(s,\mathbf{\Omega}) \tag{1}$$

where I is the radiation intensity, β the extinction coefficient defined as the sum of the scattering coefficient σ and the absorption coefficient κ, $\mathbf{\Omega}$ the direction determined by the polar angle θ and the azimuthal angle ϕ, s an arbitrary path from the boundary to a location in the medium for a general three-dimensional medium shown in Fig. 1, and S the source function defined by

$$S(s,\mathbf{\Omega}) = (1-\omega)I_b[T(s)] + \frac{\omega}{4\pi}\int_{4\pi} P(\mathbf{\Omega}',\mathbf{\Omega})I(s,\mathbf{\Omega}')\,\mathrm{d}\Omega' \tag{2}$$

Here, ω is the albedo defined as the scattering coefficient divided by the extinction coefficient, and I_b is the Planck function for emission. Because linearly anisotropic scattering is of popular interest,[2,8,9] the phase function is chosen to be

$$P(\theta',\phi',\theta,\phi) = 1 + a_1[\cos\theta\cos\theta' + \sin\theta\cos\phi\sin\theta'\cos\phi' + \sin\theta\sin\phi\sin\theta'\sin\phi'] \tag{3}$$

where a_1 is a constant.

The integration of Eq. (1) along the physical path gives

$$I(s,\mathbf{\Omega}) = I_i(s_i,\mathbf{\Omega})\exp[-\beta(s-s_i)] + \int_{s_i}^{s} \beta S(s',\mathbf{\Omega})\exp[-\beta(s-s')]\,\mathrm{d}s' \tag{4}$$

where $I_i(s_i,\mathbf{\Omega})$ is the given intensity at the boundary. Substitution of Eqs. (3) and (4) into Eq. (2) produces

$$S(s,\mathbf{\Omega}) = (1-\omega)I_b[T(s)] + (\omega/4\pi)\,[h(s) + a_1\cos\theta h_n(s) + a_1\sin\theta\cos\phi h_l(s) + a_1\sin\theta\sin\phi h_m(s) + j(s) + a_1\cos\theta j_n(s) + a_1\sin\theta\cos\phi j_l(s) + a_1\sin\theta\sin\phi j_m(s)] \tag{5}$$

where the moments of the intensity contributed by the boundary are defined as

$$h(s) = \int_{4\pi} I_i(s_i,\mathbf{\Omega}')\exp[-\beta(s-s_i)]\,\mathrm{d}\Omega' \tag{6}$$

$$h_n(s) = \int_{4\pi} \cos\theta' I_i(s_i,\mathbf{\Omega}')\exp[-\beta(s-s_i)]\,\mathrm{d}\Omega' \tag{7}$$

$$h_l(s) = \int_{4\pi} \sin\theta'\cos\phi' I_i(s_i,\mathbf{\Omega}')\exp[-\beta(s-s_i)]\,\mathrm{d}\Omega' \tag{8}$$

$$h_m(s) = \int_{4\pi} \sin\theta'\sin\phi' I_i(s_i,\mathbf{\Omega}')\exp[-\beta(s-s_i)]\,\mathrm{d}\Omega' \tag{9}$$

and the moments of the intensity contributed by the medium are defined as

$$j(s) = \int_{4\pi}\int_{s_i}^{s} \beta S(s',\mathbf{\Omega}')\exp[-\beta(s-s')]\,\mathrm{d}s'\,\mathrm{d}\Omega' \tag{10}$$

$$j_n(s) = \int_{4\pi}\cos\theta'\int_{s_i}^{s} \beta S(s',\mathbf{\Omega}')\exp[-\beta(s-s')]\,\mathrm{d}s'\,\mathrm{d}\Omega' \tag{11}$$

$$j_l(s) = \int_{4\pi}\sin\theta'\cos\phi'\int_{s_i}^{s} \beta S(s',\mathbf{\Omega}')\exp[-\beta(s-s')]\,\mathrm{d}s'\,\mathrm{d}\Omega' \tag{12}$$

$$j_m(s) = \int_{4\pi}\sin\theta'\sin\phi'\int_{s_i}^{s} \beta S(s',\mathbf{\Omega}')\exp[-\beta(s-s')]\,\mathrm{d}s'\,\mathrm{d}\Omega' \tag{13}$$

Using Viskanta's[10] procedure, the integrals in Eqs. (6–9) are transformed into surface integrals, and the integrals in Eqs. (10–13) are transformed into volume integrals. The resultant equations are

$$h(\boldsymbol{r}) = \int_A I_i\left(\boldsymbol{r}_i, \frac{\boldsymbol{r}-\boldsymbol{r}_i}{|\boldsymbol{r}-\boldsymbol{r}_i|}\right)\exp(-\beta|\boldsymbol{r}-\boldsymbol{r}_i|)[(\boldsymbol{r}-\boldsymbol{r}_i)\cdot\boldsymbol{n}]\frac{\mathrm{d}A_i}{|\boldsymbol{r}-\boldsymbol{r}_i|^3} \tag{14}$$

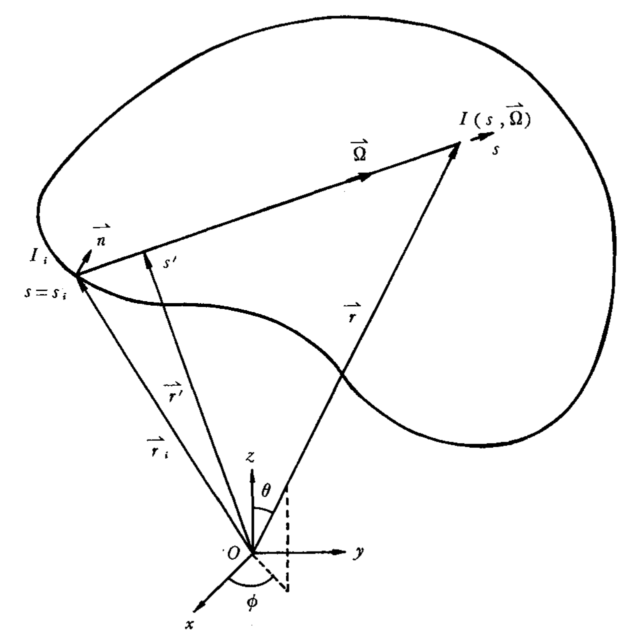

Fig. 1 Radiation intensity along a path in a general three-dimensional geometry.

$$h_n(\boldsymbol{r}) = \int_A I_i\left(\boldsymbol{r}_i, \frac{\boldsymbol{r}-\boldsymbol{r}_i}{|\boldsymbol{r}-\boldsymbol{r}_i|}\right) \exp(-\beta|\boldsymbol{r}-\boldsymbol{r}_i|) \times [(\boldsymbol{r}-\boldsymbol{r}_i)\cdot\boldsymbol{k}]\,[\boldsymbol{r}-\boldsymbol{r}_i)\cdot\boldsymbol{n}] \frac{\mathrm{d}A_i}{|\boldsymbol{r}-\boldsymbol{r}_i|^4} \tag{15}$$

$$h_l(\boldsymbol{r}) = \int_A I_i\left(\boldsymbol{r}_i, \frac{\boldsymbol{r}-\boldsymbol{r}_i}{|\boldsymbol{r}-\boldsymbol{r}_i|}\right) \exp(-\beta|\boldsymbol{r}-\boldsymbol{r}_i|) \times [(\boldsymbol{r}-\boldsymbol{r}_i)\cdot\boldsymbol{i}]\,[(\boldsymbol{r}-\boldsymbol{r}_i)\cdot\boldsymbol{n}] \frac{\mathrm{d}A_i}{|\boldsymbol{r}-\boldsymbol{r}_i|^4} \tag{16}$$

$$h_m(\boldsymbol{r}) = \int_A I_i\left(\boldsymbol{r}_i, \frac{\boldsymbol{r}-\boldsymbol{r}_i}{|\boldsymbol{r}-\boldsymbol{r}_i|}\right) \exp(-\beta|\boldsymbol{r}-\boldsymbol{r}_i|) \times [(\boldsymbol{r}-\boldsymbol{r}_i)\cdot\boldsymbol{j}]\,[(\boldsymbol{r}-\boldsymbol{r}_i)\cdot\boldsymbol{n}] \frac{\mathrm{d}A_i}{|\boldsymbol{r}-\boldsymbol{r}_i|^4} \tag{17}$$

$$j(\boldsymbol{r}) = \int_V \beta S\left(\boldsymbol{r}', \frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|}\right) \exp(-\beta|\boldsymbol{r}-\boldsymbol{r}'|) \frac{\mathrm{d}V'}{|\boldsymbol{r}-\boldsymbol{r}'|^2} \tag{18}$$

$$j_n(\boldsymbol{r}) = \int_V \beta S\left(\boldsymbol{r}', \frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|}\right) \exp(-\beta|\boldsymbol{r}-\boldsymbol{r}'|)[(\boldsymbol{r}-\boldsymbol{r}')\cdot\boldsymbol{k}] \frac{\mathrm{d}V'}{|\boldsymbol{r}-\boldsymbol{r}'|^3} \tag{19}$$

$$j_l(\boldsymbol{r}) = \int_V \beta S\left(\boldsymbol{r}', \frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|}\right) \exp(-\beta|\boldsymbol{r}-\boldsymbol{r}'|)[(\boldsymbol{r}-\boldsymbol{r}')\cdot\boldsymbol{i}] \frac{\mathrm{d}V'}{|\boldsymbol{r}-\boldsymbol{r}'|^3} \tag{20}$$

and

$$j_m(\boldsymbol{r}) = \int_V \beta S\left(\boldsymbol{r}', \frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|}\right) \exp(-\beta|\boldsymbol{r}-\boldsymbol{r}'|)\,[(\boldsymbol{r}-\boldsymbol{r}')\cdot\boldsymbol{j}] \frac{\mathrm{d}V'}{|\boldsymbol{r}-\boldsymbol{r}'|^3} \tag{21}$$

where the vector notations are shown in Fig. 1, and $\boldsymbol{n}$ is the inward normal at the boundary point $\boldsymbol{r}_i$. If the boundary surface is opaque, the surface integration in Eqs. (14–17) and the volume integration in Eqs. (18–21), respectively, extend over the boundary and the medium that allow the observation point to "see" the source point. This result has been noted by Garelis, Rudy, and Hickman[11] and later by Lin.[7] For a concave medium enclosed by transparent boundaries, the effects of re-entrant radiation have yet to be accounted for.

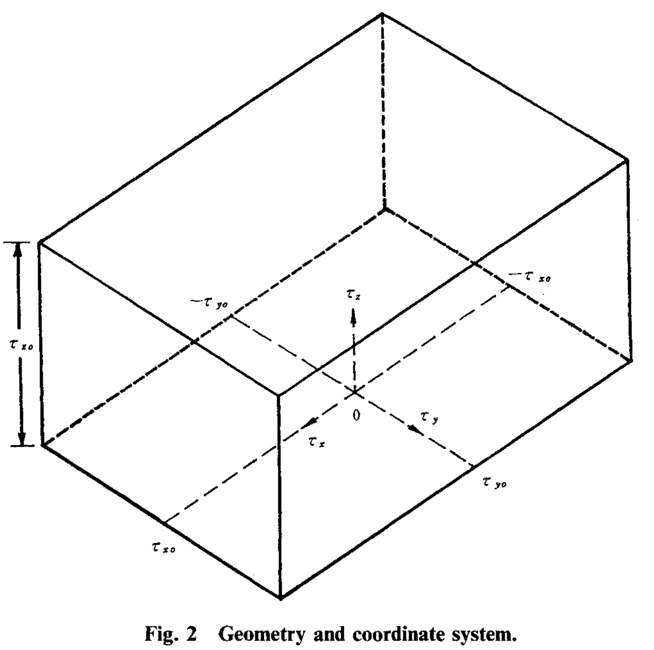

Fig. 2 Geometry and coordinate system.

Equation (5) may be written in terms of vectors as

$$S\left(\boldsymbol{r}, \frac{\boldsymbol{r}'-\boldsymbol{r}}{|\boldsymbol{r}'-\boldsymbol{r}|}\right) = (1-\omega)I_b[T(\boldsymbol{r})] + \frac{\omega}{4\pi}\left[h(\boldsymbol{r}) + a_1\frac{(\boldsymbol{r}'-\boldsymbol{r})\cdot\boldsymbol{k}}{|\boldsymbol{r}'-\boldsymbol{r}|}h_n(\boldsymbol{r}) + a_1\frac{(\boldsymbol{r}'-\boldsymbol{r})\cdot\boldsymbol{i}}{|\boldsymbol{r}'-\boldsymbol{r}|}h_l(\boldsymbol{r}) + a_1\frac{(\boldsymbol{r}'-\boldsymbol{r})\cdot\boldsymbol{j}}{|\boldsymbol{r}'-\boldsymbol{r}|}h_m(\boldsymbol{r}) + j(\boldsymbol{r}) + a_1\frac{(\boldsymbol{r}'-\boldsymbol{r})\cdot\boldsymbol{k}}{|\boldsymbol{r}'-\boldsymbol{r}|}j_n(\boldsymbol{r}) + a_1\frac{(\boldsymbol{r}'-\boldsymbol{r})\cdot\boldsymbol{i}}{|\boldsymbol{r}'-\boldsymbol{r}|}j_l(\boldsymbol{r}) + a_1\frac{(\boldsymbol{r}'-\boldsymbol{r})\cdot\boldsymbol{j}}{|\boldsymbol{r}'-\boldsymbol{r}|}j_m(\boldsymbol{r})\right] \tag{22}$$

Substitution of Eq. (22) into Eqs. (18–21) produces four integral equations for the four unknowns $j(\boldsymbol{r})$, $j_n(\boldsymbol{r})$, $j_l(\boldsymbol{r})$, and $j_m(\boldsymbol{r})$, which are the functions of position $\boldsymbol{r}$ only.

Integral Equations for Three-Dimensional Rectangular Medium with Fresnel Boundaries

In this subsection, the image technique[6] is adapted to derive the integral equations for the radiative transfer in an anisotropically scattering medium with Fresnel boundaries where the radiation intensity is unknown. The system under consideration is shown in Fig. 2. The medium is of a three-dimensional rectangular geometry whose optical dimensions are $2\tau_{x0} \times 2\tau_{y0} \times \tau_{z0}$. Here, the optical coordinate system shown in Fig. 2 is defined as $\tau_x = \beta x$, $\tau_y = \beta y$, $\tau_z = \beta z$. Assumptions (1–6), listed in the previous subsection, are followed, except that the emission is neglected. Collimated radiation intensity I_c is taken to be normally incident on the top surface of the medium. The top and bottom surfaces of the medium are assumed to be optically smooth. Because the relative refractive index of the medium n may differ from unity, the reflection according to the Fresnel equation and the refraction according to Snell's law have to be taken into account at the top and bottom surfaces. The reflections produce an infinite number of images in the z direction, as shown in Fig. 3. When the other surfaces are also specularly reflecting, more images may be formed. To simplify the analysis, the lateral surfaces at $\tau_x = \pm\tau_{x0}$ and $\tau_y = \pm\tau_{y0}$ are assumed to be nonreflecting. The treatment for the radiative transfer in an isotropically scattering medium with three or more Fresnel boundaries may be found in Ref. 6.

Figure 3 shows that radiant energy may be transferred from a source point to an observation point directly or by reflection at mirrorlike surfaces. The transfer by specular reflection may be treated as the transfer from the images of the source point to the observation point, as shown in Fig. 3. Equivalently, these images of the medium may be treated as effective media.

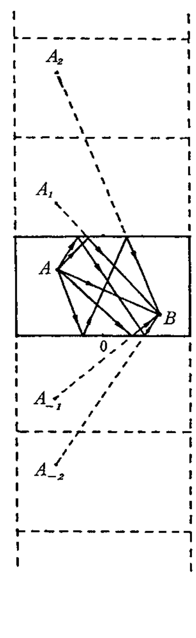

Fig. 3 Front view of a medium and its images ($A2$, A_1, A_{-1}, and A_{-2} are images of the source point A).

The radiant energy from these effective media takes the place of that transferred by reflection at Fresnel boundaries. For convenience, we use integer p to denote the location of image. As shown in Fig. 3, p is negative if the last reflection happens at the bottom surface, zero for direct transfer, and positive if the last reflection happens at the top surface. Then, the source function in the direction from point (τ_x,τ_y,τ_z) to point $(\tau'_x,\tau'_y,\tau'_z)$ in an image denoted by p can be expressed by

$$S_p(\tau_x,\tau_y,\tau_z,\tau'_x,\tau'_y,\tau'_z) = \frac{\omega}{4\pi}\Bigg[h(\tau_x,\tau_y,\tau_z) + a_1\frac{\Delta\tau_{pz}}{\Delta\tau_p}h_n(\tau_x,\tau_y,\tau_z) + a_1\frac{(\tau'_x-\tau_x)}{\Delta\tau_p}h_l(\tau_x,\tau_y,\tau_z) + a_1\frac{(\tau'_y-\tau_y)}{\Delta\tau_p}h_m(\tau_x,\tau_y,\tau_z) + j(\tau_x,\tau_y,\tau_z) + a_1\frac{\Delta\tau_{pz}}{\Delta\tau_p}j_n(\tau_x,\tau_y,\tau_z) + a_1\frac{(\tau'_x-\tau_x)}{\Delta\tau_p}j_l(\tau_x,\tau_y,\tau_z) + a_1\frac{(\tau'_y-\tau_y)}{\Delta\tau_p}j_m(\tau_x,\tau_y,\tau_z)\Bigg] \quad (23)$$

where the optical distance from point $\beta(\tau_x,\tau_y,\tau_z)$ in realistic medium to point $A(\tau'_x,\tau'_y,\tau'_z)$ in the image denoted by p is

$$\Delta\tau_p = [(\tau'_x-\tau_x)^2 + (\tau'_y-\tau_y)^2 + (\Delta\tau_{pz})^2]^{1/2} \quad (24)$$

and its z component is

$$\begin{aligned}\Delta\tau_{pz} &= (\tau_{z0}-\tau_z) + 2\tau_{z0}\{p\} - (-1)^p(\tau_{z0}-\tau'_z), \quad p>0\\ &= \tau'_x - \tau_z, \quad p=0\\ &= [\tau_z + 2\tau_{z0}\{-p\} - (-1)^{-p}\tau'_z], \quad p<0\end{aligned} \quad (25)$$

with special symbol

$$\begin{aligned}\{p\} &= p/2, \quad p = 0, 2, 4, 6, \ldots\ldots\\ &= (p-1)/2, \quad p = 1, 3, 5, 7, \ldots\ldots\end{aligned} \quad (26)$$

Since the radiation intensity from images takes the place of the reflected radiation intensity at Fresnel boundaries, the only given intensity at the boundaries is the collimated incident radiation and its reflections. For the present case, Eqs. (14–17) may be expressed as

$$h(\tau_x,\tau_y,\tau_z) = I_c(\tau_x,\tau_y)(1-\rho_n) \times \frac{\exp[-(\tau_{z0}-\tau_z)] + \rho_n\exp[-(\tau_{z0}+\tau_z)]}{1-\rho_n^2 e^{-2\tau_{z0}}} \quad (27)$$

$$h_n(\tau_x,\tau_y,\tau_z) = I_c(\tau_x,\tau_y)(1-\rho_n)\,\frac{-\exp[-(\tau_{z0}-\tau_z)] + \rho_n\exp[-(\tau_{z0}+\tau_z)]}{1-\rho_n^2 e^{-2\tau_{z0}}} \quad (28)$$

$$h_l(\tau_x,\tau_y,\tau_z) = 0 \quad (29)$$

$$h_m(\tau_x,\tau_y,\tau_z) = 0 \quad (30)$$

where $I_c(\tau_x,\tau_y)$ is the spatial variation of the incident radiation and

$$\rho_n = [(1-n)/(1+n)]^2 \quad (31)$$

Here, $h_l = 0$ and $h_m = 0$ because the collimated incident radiation is normal to the top surface.

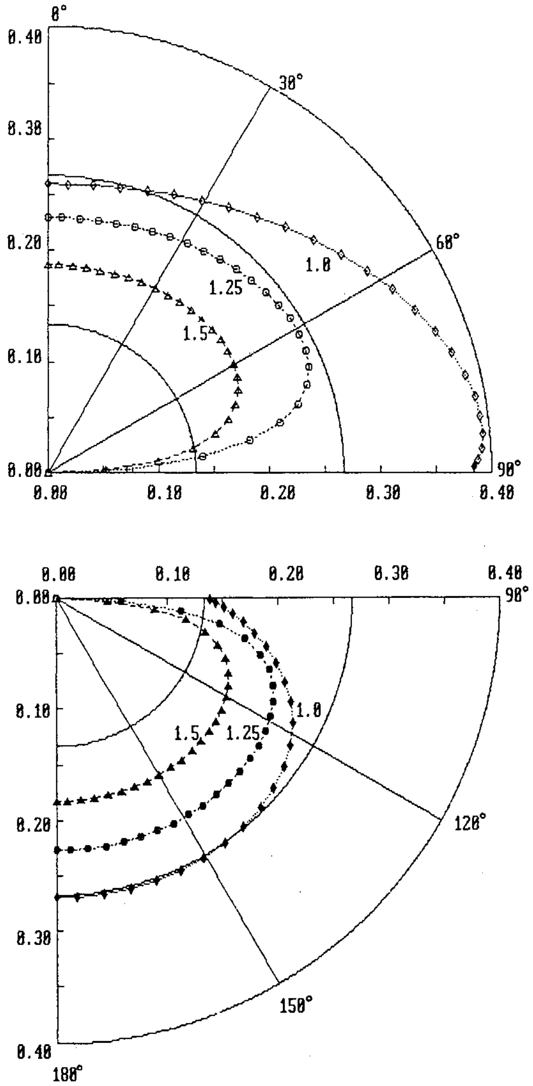

Fig. 4 Directional reflectance (top curves) and transmittance (bottom curves) calculated for various n ($\omega = 0.9$, $a_1 = 0.5$, $\tau_{z0} = 2.0$).

Equations (18–21) for the present case can be expressed as

$$j(\tau_x,\tau_y,\tau_z) = \sum_p \int_0^{\tau_{z0}}\int_{-\tau_{y0}}^{\tau_{y0}}\int_{-\tau_{x0}}^{\tau_{x0}} [\rho_p(\tau_x,\tau_y,\tau_z,\tau'_x,\tau'_y,\tau'_z)]^{|p|} \times S_p(\tau'_x,\tau'_y,\tau'_z,\tau_x,\tau_y,\tau_z)\frac{e^{-\Delta\tau_p}}{(\Delta\tau_p)^2}\,d\tau'_x\,d\tau'_y\,d\tau'_z \quad (32)$$

$$j_n(\tau_x,\tau_y,\tau_z) = \sum_p \int_0^{\tau_{z0}}\int_{-\tau_{y0}}^{\tau_{y0}}\int_{-\tau_{x0}}^{\tau_{x0}} [\rho_p(\tau_x,\tau_y,\tau_z,\tau'_x,\tau'_y,\tau'_z)]^{|p|} \times S_p(\tau'_x,\tau'_y,\tau'_z,\tau_x,\tau_y,\tau_z)\frac{e^{-\Delta\tau_p}}{(\Delta\tau_p)^3}\Delta\tau_{pz}\,d\tau'_x\,d\tau'_y\,d\tau'_z \quad (33)$$

$$j_l(\tau_x,\tau_y,\tau_z) = \sum_p \int_0^{\tau_{z0}}\int_{-\tau_{y0}}^{\tau_{y0}}\int_{-\tau_{x0}}^{\tau_{x0}} [\rho_p(\tau_x,\tau_y,\tau_z,\tau'_x,\tau'_y,\tau'_z)]^{|p|} \times S_p(\tau'_x,\tau'_y,\tau'_z,\tau_x,\tau_y,\tau_z)\frac{e^{-\Delta\tau_p}}{(\Delta\tau_p)^3}(\tau_x-\tau'_x)\,d\tau'_x\,d\tau'_y\,d\tau'_z \quad (34)$$

$$j_m(\tau_x,\tau_y,\tau_z) = \sum_p \int_0^{\tau_{z0}}\int_{-\tau_{y0}}^{\tau_{y0}}\int_{-\tau_{x0}}^{\tau_{x0}} [\rho_p(\tau_x,\tau_y,\tau_z,\tau'_x,\tau'_y,\tau'_z)]^{|p|} \times S_p(\tau'_x,\tau'_y,\tau'_z,\tau_x,\tau_y,\tau_z)\frac{e^{-\Delta\tau_p}}{(\Delta\tau_p)^3}(\tau_y-\tau'_y)\,d\tau'_x\,d\tau'_y\,d\tau'_z \quad (35)$$

where

$$\Delta\tau'_{pz} = -[(\tau_{z0}-\tau_z) + 2\tau_{z0}\{p\} - (-1)^p(\tau_{z0}-\tau'_z)], \quad p>0$$
$$= \tau_z - \tau'_z, \quad p=0$$
$$= \tau_z + 2\tau_{z0}\{-p\} - (-1)^{-p}\tau'_z, \quad p<0 \tag{36}$$

Here, the reflectivity is given by the Fresnel equation as

$$\rho_p(\tau_x,\tau_y,\tau_z,\tau'_x,\tau'_y,\tau'_z) = \frac{1}{2}\left\{\left[\frac{(1-n^2\sin^2\theta)^{1/2}-n\cos\theta}{(1-n^2\sin^2\theta)^{1/2}+n\cos\theta}\right]^2 + \left[\frac{\cos\theta - n(1-n^2\sin^2\theta)^{1/2}}{\cos\theta + n(1-n^2\sin^2\theta)^{1/2}}\right]^2\right\}, \quad \theta<\theta_{cr}$$
$$= 1, \quad \theta \geq \theta_{cr} \tag{37}$$

where

$$\theta_{cr} = \sin^{-1}(1/n) \tag{38}$$

and

$$\theta = \sin^{-1}\left\{\frac{[(\tau'_x-\tau_x)^2 + (\tau'_y-\tau_y)^2]^{1/2}}{\Delta\tau_p}\right\} \tag{39}$$

Substitution of Eqs. (23–26) into Eqs. (32–35) produces four integral equations for four unknowns j, j_n, j_l, and j_m as follows:

$$j(\tau_x,\tau_y,\tau_z) = \sum_p \int_0^{\tau_{z0}}\int_{-\tau_{y0}}^{\tau_{y0}}\int_{-\tau_{x0}}^{\tau_{x0}} [\rho_p(\tau_x,\tau_y,\tau_z,\tau'_x,\tau'_y,\tau'_z)]^{|p|} \times \left\{\frac{\omega}{4\pi}\left[h(\tau'_x,\tau'_y,\tau'_z) + a_1\frac{\Delta\tau'_{pz}}{\Delta\tau_p}h_n(\tau'_x,\tau'_y,\tau'_z) + a_1\frac{\tau_x-\tau'_x}{\Delta\tau_p}h_l(\tau'_x,\tau'_y,\tau'_z) + a_1\frac{\tau_y-\tau'_y}{\Delta\tau_p}h_m(\tau'_x,\tau'_y,\tau'_z) + j(\tau'_x,\tau'_y,\tau'_z) + a_1\frac{\Delta\tau'_{pz}}{\Delta_{\tau_p}}j_n(\tau'_x,\tau'_y,\tau'_z) + a_1\frac{\tau_x-\tau'_x}{\Delta\tau_p}j_l(\tau'_x,\tau'_y,\tau'_z) + a_1\frac{\tau_y-\tau'_y}{\Delta\tau_p}j_m(\tau'_x,\tau'_y,\tau'_z)\right]\right\}\frac{e^{-\Delta\tau_p}}{(\Delta\tau_p)^2}\,d\tau'_x\,d\tau'_y\,d\tau'_z \tag{40}$$

$$j_m(\tau_x,\tau_y,\tau_z) = \sum_p \int_0^{\tau_{z0}}\int_{-\tau_{y0}}^{\tau_{y0}}\int_{-\tau_{x0}}^{\tau_{x0}} [\rho_p(\tau_x,\tau_y,\tau_z,\tau'_x,\tau'_y,\tau'_z)]^{|p|} \times \left\{\frac{\omega}{4\pi}[h(\tau'_x,\tau'_y,\tau'_z) + a_1\frac{\Delta\tau'_{pz}}{\Delta\tau_p}h_n(\tau'_x,\tau'_y,\tau'_z) + a_1\frac{\tau_x-\tau'_x}{\Delta\tau_p}h_l(\tau'_x,\tau'_y,\tau'_z) + a_1\frac{\tau_y-\tau'_y}{\Delta\tau_p}h_m(\tau'_x,\tau'_y,\tau'_z) + j(\tau'_x,\tau'_y,\tau'_z) + a_1\frac{\Delta\tau'_{pz}}{\Delta\tau_p}j_n(\tau'_x,\tau'_y,\tau'_z) + a_1\frac{\tau_x-\tau'_x}{\Delta\tau_p}j_l(\tau'_x,\tau'_y,\tau'_z) + a_1\frac{\tau_y-\tau'_y}{\Delta\tau_p}j_m(\tau'_x,\tau'_y,\tau'_z)\right]\right\}\frac{e^{-\Delta\tau_p}}{(\Delta\tau_p)^3}\Delta\tau'_{pz}\,d\tau'_x\,d\tau'_y\,d\tau'_z \tag{41}$$

$$j_l(\tau_x,\tau_y,\tau_z) = \sum_p \int_0^{\tau_{z0}}\int_{-\tau_{y0}}^{\tau_{y0}}\int_{-\tau_{x0}}^{\tau_{x0}} [\rho_p(\tau_x,\tau_y,\tau_z,\tau'_x,\tau'_y,\tau'_z)]^{|p|} \times \left\{\frac{\omega}{4\pi}\left[h(\tau'_x,\tau'_y,\tau'_z) + a_1\frac{\Delta\tau'_{pz}}{\Delta\tau_p}h_n(\tau'_x,\tau'_y,\tau'_z) + a_1\frac{\tau_x-\tau'_x}{\Delta\tau_p}h_l(\tau'_x,\tau'_y,\tau'_z) + a_1\frac{\tau_y-\tau'_y}{\Delta\tau_p}h_m(\tau'_x,\tau'_y,\tau'_z) + j(\tau'_x,\tau'_y,\tau'_z) + a_1\frac{\Delta\tau'_{pz}}{\Delta\tau_p}j_n(\tau'_x,\tau'_y,\tau'_z) + a_1\frac{\tau_x-\tau'_x}{\Delta\tau_p}j_l(\tau'_x,\tau'_y,\tau'_z) + a_1\frac{\tau_y-\tau'_y}{\Delta\tau_p}j_m(\tau'_x,\tau'_y,\tau'_z)\right]\right\}\frac{e^{-\Delta\tau_p}}{(\Delta\tau_p)^3}(\tau_x-\tau'_x)\,d\tau'_x\,d\tau'_y\,d\tau'_z \tag{42}$$

$$j_m(\tau_x,\tau_y,\tau_z) = \sum_p \int_0^{\tau_{z0}}\int_{-\tau_{y0}}^{\tau_{y0}}\int_{-\tau_{x0}}^{\tau_{x0}} [\rho_p(\tau_x,\tau_y,\tau_z,\tau'_x,\tau'_y,\tau'_z)]^{|p|} \times \left\{\frac{\omega}{4\pi}\left[h(\tau'_x,\tau'_y,\tau'_z) + a_1\frac{\Delta\tau'_{pz}}{\Delta\tau_p}h_n(\tau'_x,\tau'_y,\tau'_z) + a_1\frac{\tau_x-\tau'_x}{\Delta\tau_p}h_l(\tau'_x,\tau'_y,\tau'_z) + a_1\frac{\tau_y-\tau'_y}{\Delta\tau_p}h_m(\tau'_x,\tau'_y,\tau'_z) + j(\tau'_x,\tau'_y,\tau'_z) + a_1\frac{\Delta\tau'_{pz}}{\Delta\tau_p}j_n(\tau'_x,\tau'_y,\tau'_z) + a_1\frac{\tau_x-\tau'_x}{\Delta\tau_p}j_l(\tau'_x,\tau'_y,\tau'_z) + a_1\frac{\tau_y-\tau'_y}{\Delta\tau_p}j_m(\tau'_x,\tau'_y,\tau'_z)\right]\right\}\frac{e^{-\Delta\tau_p}}{(\Delta\tau_p)^3}(\tau_y-\tau'_y)\,d\tau'_x\,d\tau'_y\,d\tau'_z \tag{43}$$

It is worth noting that there are no Fresnel reflections on the boundaries of a medium with unity refractive index. For such a medium, the integral terms of Eqs. (40–43) with $p \neq 0$ disappear. Comparing the resulting equations with the original equations [Eqs. (40–43)], it is readily seen that the integral terms with $p \neq 0$ make the integral equations for a medium with Fresnel boundaries different from those for a medium without reflecting boundaries. The type of the integral equations is not changed, however, by the appearance of Fresnel reflections at the boundaries.

Computational Examples and Discussion

Radiative transfer in a slab with normal incidence is governed by the one-dimensional version of Eqs. (40–43) as follows:

$$j(\tau_z) = \frac{\omega}{2}\int_0^{\tau_{z0}} [h(\tau'_z) + j(\tau'_z)]\,[F_{0000}(\tau_{z0},\tau_z,\tau'_z,n) + E_1(|\tau_z-\tau'_z|)]\,d\tau'_z + \frac{a_1\omega}{2}\int_0^{\tau_{z0}} [h_n(\tau'_z) + j_n(\tau'_z)]\,[F_{1001}(\tau_{z0},\tau_z,\tau'_z,n) + E_2(|\tau_z-\tau'_z|)\,\mathrm{sgn}(\tau_z-\tau'_z)]\,d\tau'_z \tag{44}$$

$$j_n(\tau_z) = \frac{\omega}{2}\int_0^{\tau_{z0}} [h(\tau'_z) + j(\tau'_z)]\,[F_{0101}(\tau_{z0},\tau_z,\tau'_z,n) + E_2(|\tau_z-\tau'_z|)\,\mathrm{sgn}(\tau_z-\tau'_z)]\,d\tau'_z + \frac{a_1\omega}{2}\int_0^{\tau_{z0}} [h_n(\tau'_z) + j_n(\tau'_z)]\,[F_{1100}(\tau_{z0},\tau_z,\tau'_z,n) + E_3(|\tau_z-\tau'_z|)]\,d\tau'_z \tag{45}$$

$$j_l(\tau_z) = 0 \tag{46}$$

$$j_m(\tau_z) = 0 \tag{47}$$

where

$$h(\tau_z) = I_c(1-\rho_n)\{\exp[-(\tau_{z0}-\tau_z)] + \rho_n\exp[-(\tau_{z0}+\tau_z)]\} \times [1-\rho_n^2\exp(-2\tau_{z0})]^{-1} \tag{48}$$

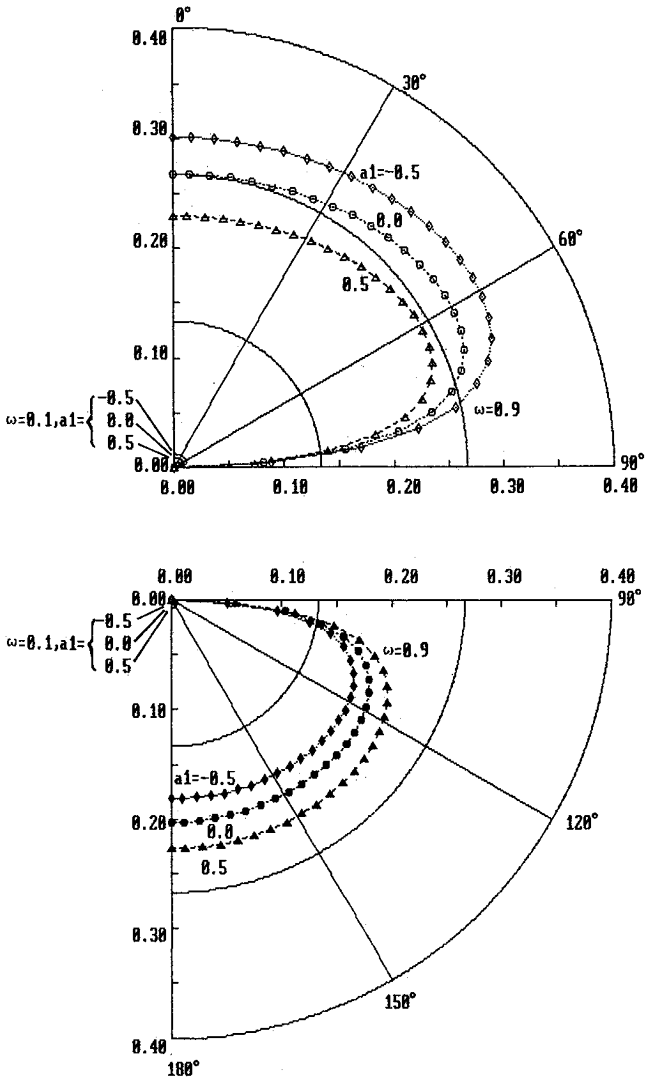

Fig. 5 Directional reflectance (top curves) and transmittance (bottom curves) calculated for various ω and a_1 ($n = 1.25$, $\tau_{z0} = 2.0$).

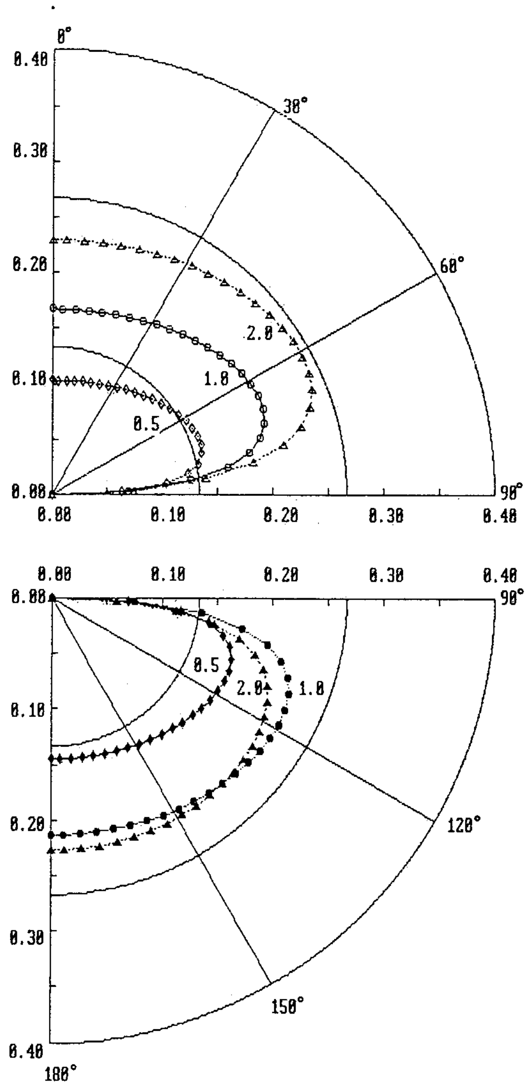

Fig. 6 Directional reflectance (top curves) and transmittance (bottom curves) calculated for various τ_{z0} ($\omega = 0.9$, $n = 1.25$, $a_1 = 0.5$).

$$h_n(\tau_z) = I_c(1-\rho_n)\{-\exp[-(\tau_{z0}-\tau_z)] + \rho_n \exp[-(\tau_{z0c}+\tau_z)]\} [1-\rho_n^2 \exp(-2\tau_{z0})]^{-1} \quad (49)$$

$$F_{qrtk}(\tau_{z0},\tau_z,\tau_z',n) = \int_0^1 \rho(\mu)\{1-[\rho(\mu)]^2 \exp(-2\tau_{z0}/\mu)\}^{-1} \times \{(-1)^q \exp[-(\tau_z+\tau_z')/\mu] + (-1)^r \exp[-(2\tau_{z0}-\tau_z-\tau_z')/\mu] + (-1)^t \rho(\mu) \exp[-(2\tau_{z0}+\tau_z-\tau_z')/\mu)] + (-1)^k \rho(\mu) \exp[-(2\tau_{z0}-\tau_z+\tau_z')/\mu]\}(\mu)^{q+r-1}\, d\mu \quad (50)$$

with

$$\mu = \cos\theta \quad (51)$$

Equations (44–45) are solved by Gaussian quadrature here. After solving $j(\tau_z)$ and $j_n(\tau_z)$, we obtain the source function from the equation

$$S(\tau_z,\mu) = \frac{\omega}{4\pi}[h(\tau_z) + a_1\mu h_n(\tau_z) + j(\tau_z) + a_1\mu j_n(\tau_x)] \quad (52)$$

Table 1 Hemispherical reflectance of an isotropically scattering slab with normal incidence

Optical thickness, τ_{z0}	Hemispherical reflectance: Integral	Hemispherical reflectance: Exact (Ref. 9)
1.0	0.3412	0.3413
2.0	0.5174	0.5175
3.0	0.6223	0.6225
4.0	0.6908	0.6909
5.0	0.7388	0.7387
6.0	0.7741	0.7738
7.0	0.8011	0.8007
8.0	0.8226	0.8218
9.0	0.8400	0.8389
10.0	0.8544	0.8530

Substituting the source function into the formal solutions[12] of radiative intensities at the top and bottom surfaces and multiplying the results by $\{[1-\rho(\mu)]/(I_c n^2)\}$, we can obtain directional reflectance and directional transmittance.

Radiative transfer in a slab with a refractive index of unity is studied first because its exact solutions exist. For this case, $\rho(\mu) = 0$ and F functions in Eqs. (44) and (45) disappear. The reflectances for various optical thicknesses are shown in Tables 1 and 2 for isotropic scattering and linearly anisotropic scattering, respectively. While a 12-point quadrature is used, present results and exact results[9] are within 0.2% for all optical thicknesses as presented in Table 1, and the range of required CPU time on CDC CYBER-830 is 0.230–0.995 s. Table 2 shows that the number of quadrature points required to obtain error within 1.0% on reflectance increases as the optical thickness increases. Results for $\tau_{z0} = 1.0$, 2.0, and 3.0 require approximately 0.80, 0.96, and 2.6 s execution, respectively, on CDC CYBER-830. Therefore, this method is economical, provided that optical thickness is not too large.

In the second case, we take the effects of the refractive index into account. A direct effect of the refractive index is Fresnel reflection at the boundaries. The specular reflection reduces the intensity of radiation leaving the slab. Thus, the directional reflectance and transmittance decrease as the refractive index increases, as shown in Fig. 4. The influence of the albedo and the anisotropic scattering is shown in Fig. 5. From Fig. 5, we know that the directional reflectance increases as a_1 decreases and that the directional transmittance decreases as a_1 decreases. The effects of anisotropic scattering can be explained as follows: radiation originating at a point in the medium or at a boundary is more easily reflected back into the surroundings when a proportionately smaller forward scattering exists. The same tendency can be found in Table 2. Because the decrease in the albedo corresponds to the increase in the absorption of radiation by the slab, the directional reflectance and transmittance for a slab with $\omega = 0.1$ are less than those for a slab with $\omega = 0.9$. Figure 6 shows the effects of the optical thickness on the radiation in a slab with $\omega = 0.9$. It is found that the directional reflectance increases as the optical thickness increases. However, it is also found that the directional transmittance does not increase monotonically as the optical thickness decreases.

Because the main objective of this work is the formulation of the radiative transfer in a three-dimensional medium with linearly anisotropic scattering and Fresnel boundaries, the three-dimensional formulation has not been solved here. However, the three-dimensional formulation obtained by this analysis will still be solvable because the numerical solutions of the triple integrals appearing in this work already exist for limited cases.[13] The formulation should be solved in future works. The one-dimensional solutions presented here give us some physical insight and an initial evaluation of present ideas. Comparing the results with existing exact solutions shows that this analysis works quite well.

Table 2 Hemispherical reflectance of a linearly anisotropically scattering slab with normal incidence

Optical thickness, τ_{z0}	No. of quadrature for integral formulation	Hemispherical reflectance: Integral	Hemispherical reflectance: Exact (Ref. 9)
	$a_1 = 0.5$		
1.0	24	0.2912	0.2924
2.0	24	0.4652	0.4654
3.0	36	0.5762	0.5743
4.0	60	0.6499	0.6473
5.0	96	0.7016	0.6997
	$a_1 = 1.0$		
1.0	24	0.2332	0.2355
2.0	24	0.3990	0.4006
3.0	36	0.5149	0.5120
4.0	60	0.5943	0.5901
5.0	96	0.6509	0.6471

Summary and Conclusions

Integral equations in terms of the moments of the intensity were developed for the radiative transfer in three-dimensional anisotropically scattering media. First, we considered the radiative transfer in an arbitrary three-dimensional medium with a given inward boundary intensity. It is shown that the present method is easily applied to arbitrary geometries. Next, we considered the radiative transfer in a three-dimensional rectangular medium with Fresnel boundaries. To take account of the effects of Fresnel reflection, the image technique was adapted for use in the present anisotropically scattering case. Finally, specific examples are solved by present methods. The results give us some physical insight and an initial evaluation of present ideas. It is found that the number of quadrature points is increased to obtain accurate results as the optical thickness increases. The solutions for multidimensional geometries should be performed. Furthermore, the integral equations should also be developed for the radiative transfer in an anisotropically scattering medium with a more general phase function and diffusely reflecting boundaries.

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